UFR ST - Université de Franche-Comté

M1 Project

A proof of Weiner's Theorem

Girod Manon & Mathieu Alix
M1 Mathématiques

Tuteur : M. Boussaid

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Nomenclature

$\mathscr{C}$ Unit circle

$\mathscr{C}(0, R_1, R_2)$ Annulus defined by an open region in complex plane such that $0 \leq R_1 < |z| < R_2$

Γ_r Circle of radius $r > 0$

$\mathscr{B}(z, \varepsilon)$ Ball of radius $\varepsilon > 0$ centered at point z

1 General statements on Banach algebras

1.1 Some reminders on Banach spaces

For this section, $(E, \|.\|)$ is a normed vector space.

Definition 1.1.1 : (Convergent sequence)
Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in E and e in E.
We say that $(e_n)_{n\in\mathbb{N}}$ **converges toward** e **in** E when : $\|e_n - e\| \underset{n\to\infty}{\longrightarrow} 0$.

Definition 1.1.2 : (Cauchy sequence and Banach space)
Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in E.

- We say that $(e_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** when :

$$\forall \varepsilon > 0, \exists n \in \mathbb{N}, \forall p, q \geq n, \|e_p - e_q\| \leq \varepsilon.$$

- We say that $(E, \|.\|)$ is a **Banach space** when any Cauchy sequence of E is convergent in E.

Definition 1.1.3 : (Convergent series and normally convergent series)
Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in E.

- We say that $\sum\limits_{n\in\mathbb{N}} e_n$ is **convergent in** E when $\left(\sum\limits_{n=0}^{N} e_n\right)_{N\in\mathbb{N}}$ converges in E.
- We say that $\sum\limits_{n\in\mathbb{N}} e_n$ is **normally convergent** when $\sum\limits_{n\in\mathbb{N}} \|e_n\|$ converges (in $\mathbb{R}$).

Theorem 1.1.4 : Characterisation of Banach spaces
Normed vector space $(E, \|.\|)$ is a Banach space if and only if any normally convergent series is convergent.

Proof :

— Let us assume that $(E, \|.\|)$ is a Banach space.
Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in E such that $\sum\limits_{n\in\mathbb{N}} e_n$ is normally convergent.

Let $k > l$. Then :

$$\begin{aligned}\left\|\sum_{n=0}^{k} e_n - \sum_{n=0}^{l} e_n\right\| &= \left\|\sum_{n=l+1}^{k} e_n\right\| \\ &\leq \sum_{n=l+1}^{k} \|e_n\| \\ &\leq \sum_{n=l+1}^{+\infty} \|e_n\|\end{aligned}$$

But, $\sum\limits_{n\in\mathbb{N}} e_n$ is normally convergent so $\sum\limits_{n\in\mathbb{N}} \|e_n\|$ converges. Hence, $\sum\limits_{n=l+1}^{+\infty} \|e_n\|$ is a remainder of a convergent series. So, $\sum\limits_{n=l+1}^{+\infty} \|e_n\| \underset{l\to\infty}{\longrightarrow} 0$.

So, $\left(\sum\limits_{n\in\mathbb{N}} e_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence is Banach space E.

Hence, $\left(\sum_{n\in\mathbb{N}} e_n\right)_{n\in\mathbb{N}}$ converges in E which means $\sum_{n\in\mathbb{N}} e_n$ converges.

— Let us assume that any normally convergent series is convergent.
Let $(e_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in E. Then,

$$\forall \varepsilon > 0, \exists n \in \mathbb{N}, \forall p, q \geq n, \|e_p - e_q\| \leq \varepsilon$$

For $\varepsilon = \left(\frac{1}{2}\right)^0 = 1 > 0, \exists n_0 \in \mathbb{N}, \forall p, q \geq n_0, \|e_p - e_q\| \leq \left(\frac{1}{2}\right)^0$.
So, in particular, $\|e_{n_0+1} - e_{n_0}\| \leq \left(\frac{1}{2}\right)^0$.
For $\varepsilon = \left(\frac{1}{2}\right)^1$, $\exists n_1 > n_0, \forall p, q \geq n_1, \|e_p - e_q\| \leq \left(\frac{1}{2}\right)^1$.
So, in particular, $\|e_{n_1+1} - e_{n_1}\| \leq \left(\frac{1}{2}\right)^1$.
Hence, by induction, we built an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that :

$$\forall k \in \mathbb{N}, \|e_{n_k+1} - e_{n_k}\| \leq \left(\frac{1}{2}\right)^k$$

But, $\left(\frac{1}{2}\right)^k$ is the term of a convergent series. So, $\sum_{k\in\mathbb{N}} \|e_{n_k+1} - e_{n_k}\|$ is convergent *i.e.* $\sum_{k\in\mathbb{N}} (e_{n_k+1} - e_{n_k})$ is normally convergent. Hence, by hypothesis, $\sum_{k\in\mathbb{N}} (e_{n_k+1} - e_{n_k})$ converges in E.
But, for all $K \in \mathbb{N}$, $\sum_{k=0}^{K} (e_{n_k+1} - e_{n_k}) = e_{n_K+1} - e_{n_0}$.

So, $e_{n_K+1} = \underbrace{e_{n_0}}_{\text{constant}} + \underbrace{\sum_{k=0}^{K} (e_{n_k+1} - e_{n_k})}_{\text{term of a convergent sequence}}$. Hence, $(e_{n_K+1})_{K\in\mathbb{N}}$ is convergent in E.

So, $(e_n)_{n\in\mathbb{N}}$ is a Cauchy sequence which admits a convergent subsequence. So, $(e_n)_{n\in\mathbb{N}}$ converges in E.
Hence, we showed that any Cauchy sequence of E converges in E. So,$(E, \|.\|)$ is a Banach space.

■

1.2 First definitions and properties

DEFINITION 1.2.1 : (ALGEBRA)
Let $\mathbb{K}$ be a field.
An **algebra** $\mathcal{B}$ is a $\mathbb{K}$-vectorial space $(\mathcal{B}, +, .)$ endow with an internal associative bilinear multiplication law $\times$, *i.e.*, for all $e, f, g \in \mathcal{B}$:

i) $e \times f \in \mathcal{B}$

ii) $(e \times f) \times g = e \times (f \times g) = e \times f \times g$

iii) $e \times (f + g) = e \times f + e \times g$ and $(e + f) \times g = e \times g + f \times e$

iv) $\forall \lambda, \mu \in \mathbb{K},\ (\lambda.e) \times (\mu.f) = (\lambda\mu).(\mu \times f)$

DEFINITION 1.2.2 : (NORMED ALGEBRA)
We say that an algebra $\mathcal{B}$ is **normed** if we can endow $\mathcal{B}$ with a norm $\|.\|$ such that :

$$\forall e, f \in \mathcal{B},\ \|ef\| \leq \|e\|\,\|f\|$$

REMARK In such an algebra, we can prove by induction that : $\forall x \in \mathcal{B}, \forall n \in \mathbb{N}, \|e^n\| \leq \|e\|^n$.

DEFINITION 1.2.3 : (BANACH ALGEBRA)
A **Banach algebra** $\mathcal{B}$ is a normed algebra $(\mathcal{B}, \|.\|)$ over $\mathbb{K}$ and a Banach space endowed with the metric induced by the norm $\|.\|$.

DEFINITION 1.2.4 : (UNIT ELEMENT)
A **unit element** 1 of $\mathcal{B}$ is an element such that for all $e \in \mathcal{B},\ e1 = 1e = e$ and verifying $\|1\| = 1$.

REMARK An algebra doesn't necessary have a unit but if it exists, then it is unique.

NOTATION We will write in the remainder of the document, for all $\lambda \in \mathbb{K},\ \lambda$ for $\lambda 1$.

DEFINITION 1.2.5 : (INVERTIBLE)
An element $e \in \mathcal{B}$ is said **invertible** if there exists $f \in \mathcal{B}$ such that $ef = fe = 1$.
f is unique and will be denoted e^{-1}.

REMARK The set of invertible elements of $\mathcal{B}$ endowed with $\times$ is a multiplicative group denoted $G(\mathcal{B})$.

PROPOSITION 1.2.6 :
Let $e \in \mathcal{B}$. If $\|e\| < 1$ then

$$1 - e \text{ is invertible and } (1-e)^{-1} = \sum_{i=0}^{\infty} e^i$$

PROOF :
For $N \in \mathbb{N}$, let $S_N = \sum_{n=0}^{N} e^n$. As $\mathcal{B}$ is a Banach algebra, we have for all n in $\mathbb{N}$, $\|e^n\| \leq \|e\|^n$.But, we supposed that $\|e\| < 1$. So, $\sum_{n\in\mathbb{N}} \|e\|^n$ is convergent. So is $\sum_{n\in\mathbb{N}} \|e^n\|$. Then, $\sum_{n\in\mathbb{N}} e^n$ is normally convergent. Hence, as $\mathcal{B}$ is a Banach space, by proposition 1.1.4, $\sum_{n\in\mathbb{N}} e^n$ converges in $\mathcal{B}$.

Let $S = \sum_{n=0}^{\infty} e^n$ be its limit.
But, as sums of powers of e, we have : $\forall N \in \mathbb{N}, S_N(1-e) = (1-e)S_N = 1 - e^{N+1}$.
So, $S(1-e) = (1-e)S = \lim_{N\to+\infty} (1 - e^{N+1}) = 1$.

Hence, $1 - e$ is invertible and $(1-e)^{-1} = S = \sum_{n=0}^{\infty} e^n$ ■

REMARK We also have if $\|e\| < 1$, that $1 + e$ invertible and $(1+e)^{-1} = \sum_{i=0}^{\infty} (-1)^i e^i$

DEFINITION 1.2.7 : (RESOLVENT SET)
The **resolvent set** of $e \in \mathcal{B}$ denoted $\rho(e)$ is defined by :

$$\rho(e) = \{\zeta \in \mathbb{C},\ e - \zeta \text{ is invertible}\}$$

DEFINITION 1.2.8 : (RESOLVENT MAP)
The **resolvent** of $e \in \mathcal{B}$ is defined by the following map :

$$R_e : \left| \begin{array}{ccc} \rho(e) & \longrightarrow & G(\mathcal{B}) \\ \zeta & \longmapsto & (e-\zeta)^{-1} \end{array} \right.$$

DEFINITION 1.2.9 : (SPECTRUM)
The **spectrum** of $e \in \mathcal{B}$ is the complementary set in the complex plane of the resolvent set of e. It will be denoted :
$$\sigma(e) := \mathbb{C} \backslash \rho(e)$$

1.3 Properties of the resolvent

PROPOSITION 1.3.1 :
Let $e \in \mathcal{B}$. Then, the set $\rho(e)$ is open in $\mathbb{C}$.
Moreover, the resolvent $R_e : z \in \rho(e) \mapsto (e - z)^{-1} \in G(\mathcal{B})$ is analytic.

PROOF :
Let $a \in \rho(e)$, we have for all $z \in \mathbb{C}$:
$$\begin{aligned} e - z &= e - a + a - z \\ &= (e - a)(1 - (e - a)^{-1}(z - a)) \end{aligned}$$

If $|z - a| < \frac{1}{\|(e-a)^{-1}\|}$, then by proposition 1.2.6, $1 - (e - a)^{-1}(z - a)$ is invertible and so is $e - z$. So, $\mathscr{B}(a, \frac{1}{\|(e-a)^{-1}\|}) \subseteq \rho(e)$. Then, the resolvent set $\rho(e)$ is open. Moreover, the proposition also states that
$$\forall z \in \mathscr{B}(a, \frac{1}{\|(e-a)^{-1}\|}), (e - z)^{-1} = (e - a)^{-1} \left(\sum_{n \geq 0} ((z - a)(e - a)^{-1})^n \right)$$

This shows the resolvent map is analytic. ■

PROPOSITION 1.3.2 :
Let $e \in \mathcal{B}$.
The resolvent map $R_e : z \in \rho(e) \mapsto (e - z)^{-1} \in G(\mathcal{B})$ is holomorphic on $\rho(e)$.

PROOF :
Let $z_0 \in \rho(e)$. We have :
$$\begin{aligned} (z - z_0)^{-1}[R_e(z) - R_e(z_0)] &= (z - z_0)^{-1}[(e - z)^{-1} - (e - z_0)^{-1}] \\ &= (z - z_0)^{-1}(e - z)^{-1}(e - z_0)^{-1}[e - z_0 - e + z] \\ &= (z - z_0)^{-1}(e - z)^{-1}(e - z_0)^{-1}(z - z_0) \\ &= (e - z)^{-1}(e - z_0)^{-1} \\ &\xrightarrow[z \to z_0]{\|\cdot\|} (e - z_0)^{-2} \text{ by continuity (from analyticity 1.3.1) of the resolvent of } e \end{aligned}$$

■

1.4 Properties of a particular radius

Here is a little reminder :

PROPOSITION 1.4.1 : ROOT TEST
Let $(e_n)_{n \in \mathbb{N}} \subseteq \mathcal{B}$. Let $C := \overline{\lim_{n \to +\infty}} \sqrt[n]{\|e_n\|}$

i) If $C < 1$, the series normally converges (so converges if the space is a Banach).

ii) If $C > 1$, the series diverges.

Proof :

i) Let us assume that $C < 1$. Let $q \in \mathbb{R}$ such that $C < q < 1$. As $C < q$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N, \|e_n\|^{1/n} \leq q$. Then $(\|e_n\|)_{n\geq N}$ is upper bounded by a geometric sequence $(q^n)_{n\geq N}$. But, as $q < 1$, $\sum\limits_{n\in\mathbb{N}} q^n$ converges, so, by comparison, $\sum\limits_{n\in\mathbb{N}} \|e_n\|$ also converges.

ii) Let us assume that $C > 1$. Then, there exists an infinite number of n such that $\|e_n\|^{1/n} \geq 1$, *i.e.* $\|e_n\| \geq 1$. So $(\|e_n\|)_{n\in\mathbb{N}}$ does not converge to 0, so neither does $(e_n)_{n\in\mathbb{N}}$ and then the series diverges.

■

Proposition 1.4.2 :
Let $e \in \mathcal{B}$. The sequence $(\|e^n\|^{\frac{1}{n}})_{n\in\mathbb{N}^*}$ converges. We will call $r(e)$ this limit.

Proof :
For all $n \in \mathbb{N}^*$, let $u_n = \|e^n\|^{\frac{1}{n}}$.

* As a norm, we have for all $n \in \mathbb{N}^*$, $u_n \geq 0$
* Let us show that $(u_n)_{n\in\mathbb{N}^*}$ is a non increasing sequence :

$$\begin{aligned}
\frac{u_{n+1}}{u_n} &= \frac{\|e^{n+1}\|^{\frac{1}{n+1}}}{\|e^n\|^{\frac{1}{n}}} \\
&= \frac{\|e^n e\|^{\frac{1}{n+1}}}{\|e^n\|^{\frac{1}{n}}} \\
&\leq \frac{\|e^n\|^{\frac{1}{n+1}}\,\|e\|^{\frac{1}{n+1}}}{\|e^n\|^{\frac{1}{n}}} \\
&\leq \|e^n\|^{\frac{-1}{n(n+1)}}\,\|e\|^{\frac{1}{n+1}} \\
&\leq \|e\|^{\frac{-1}{n+1}}\,\|e\|^{\frac{1}{n+1}} \\
&\leq 1
\end{aligned}$$

So, $(u_n)_{n\in\mathbb{N}^*}$ is a non increasing sequence and lower bounded by 0 so it converges.

■

<u>Remark</u> As $(\|e^n\|^{\frac{1}{n}})_{n\in\mathbb{N}^*}$ is a non increasing sequence, we have :

$$r(e) = \lim_{n\to+\infty} \|e_n\|^{1/n} = \inf_{n\in\mathbb{N}^*} \|e_n\|^{1/n}$$

And so, $r(e) \leq \|e\|$

Lemma 1.4.3 :
Let $e \in \mathcal{B}$ and $z \in \mathbb{C}$. Then, $r(ze) = |z|\, r(e)$.

Proof :
For all $n \in \mathbb{N}$, $\|(ze)^n\| = \|z^n e^n\| = |z|^n \|e^n\|$ so $\|(ze)^n\|^{\frac{1}{n}} = |z|\,\|e^n\|^{\frac{1}{n}}$.
Hence,

$$\lim_{n\to\infty} \|(ze)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} |z|\,\|e^n\|^{\frac{1}{n}} \text{ i.e. } r(ze) = |z|\, r(e)$$

■

Proposition 1.4.4 :

i) If $r(e) < 1$, the series $\sum\limits_{n\in\mathbb{N}} e^n$ is normally convergent.

ii) If $r(e) > 1$, the series $\sum\limits_{n\in\mathbb{N}} e^n$ is divergent.

Proof :
As $r(e) = \lim\limits_{n\to+\infty} \|e^n\|^{\frac{1}{n}}$, then $\overline{\lim\limits_{n\in\mathbb{N}}} \|e^n\|^{\frac{1}{n}} = r(e)$. So, if :

i) $r(e) < 1$, by the root test, $\sum\limits_{n\in\mathbb{N}} \|e^n\|$ converges, so the series $\sum\limits_{n\in\mathbb{N}} e^n$ is normally convergent

ii) $r(e) > 1$, by the root test, the series $\sum\limits_{n\in\mathbb{N}} e^n$ is divergent.

■

Lemma 1.4.5 :
Let $e \in \mathcal{B}$. If $r(e) < 1$, then $1 - e$ is invertible. Moreover,

$$(1-e)^{-1} = \sum_{n=0}^{+\infty} e^n$$

Proof :
As $r(e) < 1$, there exists $t \in \mathbb{R}$ such that $r(e) < t < 1$.
Hence, as $\left(\|e^n\|^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ converges towards $r(e)$, $\exists N \in \mathbb{N}^*, \forall n \geq N, \|e^n\|^{\frac{1}{n}} \leq t$ i.e. $\|e^n\| \leq t^n$.
But, $\sum\limits_{n\in\mathbb{N}} t^n$ converges because $t < 1$. So, by comparison, $\sum\limits_{n\in\mathbb{N}} \|e^n\|$ converges.
Hence, $\sum\limits_{n\in\mathbb{N}} e^n$ converges normally. Moreover, as $\mathcal{B}$ is a Banach space, $\sum\limits_{n\in\mathbb{N}} e^n$ converges.

Now, let us show that $S := \sum\limits_{n=0}^{\infty} e^n$ is the inverse of e.

For all $N \in \mathbb{N}$, let $S_N := \sum\limits_{n=0}^{N} e^n$. Then, we have :

$$(1-e)S = \lim_{N\to+\infty} (1-e)S_N = \lim_{N\to+\infty} (1-e^{N+1}) = 1$$

Hence, $(1-e)S = S(1-e) = 1$. So :

$$1 - e \text{ is invertible and } (1-e)^{-1} = \sum_{n=0}^{+\infty} e^n$$

■

Proposition 1.4.6 :
If $e \in G(\mathcal{B})$ then $(e-z)^{-1}$ is a limit of a normally convergent series on $\mathscr{B}(0, \frac{1}{r(e^{-1})})$.

Proof :
We have $e - z = e(1 - ze^{-1})$. Moreover, by lemma 1.4.3, $r(ze^{-1}) = |z|\, r(e^{-1})$.
If $|z| < \frac{1}{r(e^{-1})}$, then $r(ze^{-1}) < 1$.

So, by proposition 1.4.5, $1 - ze^{-1}$ is invertible and so $(1 - ze^{-1})^{-1} = \sum\limits_{n=0}^{\infty} (ze^{-1})^n$. Moreover, proposition 1.4.4, this series is normally convergent on $\mathscr{B}(0, \frac{1}{r(e^{-1})})$. ■

PROPOSITION 1.4.7 :
Let $e \in G(\mathcal{B})$. Then, $d(0, \sigma(e)) = \frac{1}{r(e^{-1})}$.

PROOF :
By proposition 1.3.1, we know that R is analytic on $\rho(e)$. As $e \in G(\mathcal{B})$, $e = e - 0$ is invertible. So, $0 \in \rho(e)$. Let r be the radius of convergence of the power series of R at the point 0.
But, by Cauchy's theorem, we know that R is analytic on the largest disc $\mathcal{B}(0, r)$ included in its holomorphic domain. But, by proposition 1.3.2, we know that R is holomorphic on its domain of definition $\rho(e)$. So $r = d(0, \sigma(e))$.
Moreover, for all z in $\mathbb{C}$, $e - z = e(1 - ze^{-1})$.
So, for all z in $\mathbb{C}$, $e - z$ is invertible if and only if $1 - ze^{-1}$ is invertible.
Hence, for all $\Omega \subseteq \mathbb{C}$ non empty open set, we have :

$$z \mapsto (e-z)^{-1} \text{ is analytic on } \Omega \text{ if and only if } z \mapsto (1-ze^{-1})^{-1} \text{ is analytic on } \Omega$$

But, in the proof of 1.4.6, we showed that $(e-z)^{-1} = \sum\limits_{n=0}^{+\infty} (e^{-1})^{n+1} z^n$ if $z \in \mathscr{B}(0, \frac{1}{r(e^{-1})})$. But, by the root test, we also know that if $|z| > \frac{1}{r(e^{-1})}$, then the series $\sum\limits_{n \in \mathbb{N}} (e^{-1})^{n+1} z^n$ is divergent.Then, the radius of the power series of $(1 - ze^{-1})^{-1}$ at the point 0 is $\frac{1}{r(e^{-1})}$. But, by the equivalence written above, this radius is also equal to r.
Hence, $r = \frac{1}{r(e^{-1})}$ *i.e.* $d(0, \sigma(e)) = \frac{1}{r(e^{-1})}$.

■

For now, we consider $\mathcal{B}$ as a commutative Banach algebra.

PROPOSITION 1.4.8 :
Let $e, f \in \mathcal{B}$. Then :

$$r(ef) \leq r(e)r(f)$$

PROOF :
Let $n \in \mathbb{N}$. Then, because $\mathcal{B}$ is a commutative Banach algebra :

$$\|(ef)^n\| = \|e^n f^n\| \leq \|e^n\| \, \|f^n\|$$

So, $\|(ef)^n\|^{\frac{1}{n}} \leq \|e^n\|^{\frac{1}{n}} \, \|f^n\|^{\frac{1}{n}}$. Hence, $r(ef) \leq r(e)r(f)$ ■

1.5 Properties of the spectum

PROPOSITION 1.5.1 :
Let $e \in \mathcal{B}$. Let $(e_n)_{n \in \mathbb{N}} \in \mathcal{B}^{\mathbb{N}}$ be a sequence such that $e_n \xrightarrow[n \to \infty]{\|\cdot\|} e$.
If $z \in \bigcap\limits_{n \in \mathbb{N}} \overline{\bigcup\limits_{k \geq n} \sigma(e_k)}$ then $z \in \sigma(e)$. Hence, $\bigcap\limits_{n \in \mathbb{N}} \overline{\bigcup\limits_{k \geq n} \sigma(e_k)} \subseteq \sigma(e)$.

PROOF :
Let assume that $z \in \rho(e) = \mathbb{C} \backslash \sigma(e)$. Then, there exists $\varepsilon_1 > 0, \mathscr{B}(z, \varepsilon_1) \subseteq \rho(e)$.
Let $\varepsilon_2 = \frac{1}{3\|(e-z)^{-1}\|}$. As $e_k \xrightarrow[k \to \infty]{\|\cdot\|} e$, there exists $n \geq 0$ such that $\forall k \geq n, \|e_k - e\| \leq \varepsilon_2$.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then, for all w in $\mathscr{B}(z, \varepsilon)$,

$$\begin{aligned} e_k - w &= \underbrace{(e_k - e)}_{\|.\| \le \varepsilon} + \underbrace{(e - z)}_{\text{invertible}} + \underbrace{(z - w)}_{\|.\| < \varepsilon} \\ &= (e - z)(1 + (e - z)^{-1}((e_k - e) + (z - w))) \end{aligned}$$

But,

$$\begin{aligned} \left\|(e - z)^{-1}((e_k - e) + (z - w))\right\| &\le \left\|(e - z)^{-1}\right\| \|((e_k - e) + (z - w))\| \\ &\le \left\|(e - z)^{-1}\right\| 2\varepsilon \\ &\le \frac{2}{3} < 1 \end{aligned}$$

So, by proposition 1.2.6, there exists $n \ge 0$, such that $\forall k \ge n$, $e_k - w$ is invertible *i.e.* $w \in \rho(e_k)$.
So, there exists $n \ge 0$, such that $\forall k \ge n$, $\mathscr{B}(z, \varepsilon) \subseteq \rho(e_k)$, so, $\mathscr{B}(z, \varepsilon) \subseteq \bigcap\limits_{k \ge n} \rho(e_k)$.

So, there exists $n \ge 0$ such that $z \in \overbrace{\bigcap\limits_{k \ge n} \rho(e_k)}^{\circ} = \overbrace{\mathbb{C} \setminus \bigcup\limits_{k \ge n} \sigma(e_k)}^{\circ} = \mathbb{C} \setminus \overline{\bigcup\limits_{k \ge n} \sigma(e_k)}$.

Hence, $z \in \bigcup\limits_{n \in \mathbb{N}} \mathbb{C} \setminus \overline{\bigcup\limits_{k \ge n} \sigma(e_k)}$, *i.e.* $z \notin \bigcap\limits_{n \in \mathbb{N}} \overline{\bigcup\limits_{k \ge n} \sigma(e_k)}$. ■

Proposition 1.5.2 :

Let $e \in \mathcal{B}$. Let $(e_n)_{n \in \mathbb{N}} \in \mathcal{B}^{\mathbb{N}}$ be a sequence such that $e_n \xrightarrow[n \to \infty]{\|.\|} e$.
If $z \notin \bigcap\limits_{n \in \mathbb{N}} \overline{\bigcup\limits_{k \ge n} \sigma(e_k)}$ then $z \notin \sigma(e)$. Hence, $\sigma(e) \subseteq \bigcap\limits_{n \in \mathbb{N}} \overline{\bigcup\limits_{k \ge n} \sigma(e_k)}$.

Proof :

Let $z \notin \bigcap\limits_{n \in \mathbb{N}} \overline{\bigcup\limits_{k \ge n} \sigma(e_k)}$.

Hence, $z \in \bigcup\limits_{n \in \mathbb{N}} \mathbb{C} \setminus \overline{\bigcup\limits_{k \ge n} \sigma(e_k)} = \bigcup\limits_{n \in \mathbb{N}} \overbrace{\bigcap\limits_{k \ge n} \mathbb{C} \setminus \sigma(e_k)}^{\circ} = \bigcup\limits_{n \in \mathbb{N}} \overbrace{\bigcap\limits_{k \ge n} \rho(e_k)}^{\circ}$.

So, $\exists n_0 \in \mathbb{N}$ such that $z \in \overbrace{\bigcap\limits_{k \ge n_0} \rho(e_k)}^{\circ}$. Then, $\exists \varepsilon > 0$ such that $\mathscr{B}(z, \varepsilon) \subseteq \bigcap\limits_{k \ge n_0} \rho(e_k)$.
Hence, for all $k \ge n_0$, $e_k - z$ is invertible.
Moreover, as $\mathscr{B}(z, \varepsilon) \subseteq \bigcap\limits_{k \ge n_0} \rho(e_k), \forall k \ge n_0, d(0, \sigma((e_k - z)^{-1})) \ge \varepsilon$.

As $e_n \xrightarrow[n \to \infty]{\|.\|} e$, $\exists N \ge n_0$ such that $\forall n \ge N, \|e_n - e\| \le \frac{\varepsilon}{2}$.
So, we have :

$$\begin{aligned} r((e_N - z)^{-1}(e_N - e)) &\overset{\substack{1.4.8 \\ \downarrow}}{\le} r((e_N - z)^{-1}) r(e_N - e) \\ &\le \frac{1}{d(0, \sigma((e_N - z)^{-1}))} \|e_N - e\| \\ &\le \frac{\varepsilon/2}{\varepsilon} = \frac{1}{2} \\ &< 1. \end{aligned}$$

So, by proposition 1.4.5, $1 + (e_N - z)^{-1}(e_N - e)$ is invertible.
But, $e - z = e - e_N + e_N - z = ((e_N - z)^{-1}(e - e_N) + 1)(e_N - z)$.
So, as $e_N - z$ is also invertible, $e - z$ is invertible.
Hence, $z \notin \sigma(e)$. So the contrapositive gives : $z \in \sigma(e) \Rightarrow z \in \bigcap\limits_{n \in \mathbb{N}} \overline{\bigcup\limits_{k \ge n} \sigma(e_k)}$.

Then, we showed $\sigma(e) \subseteq \bigcap\limits_{n\in\mathbb{N}} \overline{\bigcup\limits_{k\geq n} \sigma(e_k)}$. ■

Corollary 1.5.3 :

Let $e \in \mathcal{B}$. Let $(e_n)_{n\in\mathbb{N}} \in \mathcal{B}^{\mathbb{N}}$ be a sequence such that $e_n \xrightarrow[n\to\infty]{\|\cdot\|} e$. Then :

$$\sigma(e) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \sigma(e_k)}.$$

2 Application

2.1 Some reminders on Laurent series

Lemma 2.1.1 :
Let $f : \mathscr{C}(0, R_1, R_2) \to \mathbb{C}$ holomorphic. Then

$$\forall r_1, r_2 \in]R_1, R_2[, \int_{\Gamma_{r_1}} f(z)dz = \int_{\Gamma_{r_2}} f(z)dz$$

Proof :
Let $J : R \in]R_1, R_2[\mapsto \int_{\Gamma_R} f(z)dz$.
Let g the holomorphic map defined by $\forall z \in \mathscr{C}(0, R_1, R_2), g(z) := zf(z)$.
Let $R \in]R_1, R_2[$. With the parametrization of Γ_R, $z = Re^{it}$ for $t \in [0, 2\pi]$, we have :

$$J(R) = i \int_0^{2\pi} g(Re^{it})dt$$

So, by differentiation under integral sign theorem, we have J differentiable and

$$\forall R \in]R_1, R_2[, J'(R) = i \int_0^{2\pi} e^{it} g'(Re^{it})dt = \frac{1}{R} \int_{\Gamma_R} g'(z)dz$$

But, we also have, as the integral of a derivative on a closed path,

$$\forall R \in]R_1, R_2[, \int_{\Gamma_R} g'(z)dz = 0, \text{i.e.} J'(R) = 0$$

So, J is constant. ■

Proposition 2.1.2 : Laurent series
Let $f : \mathscr{C}(0, R_1, R_2) \to \mathbb{C}$ holomorphic.
Then, there exists a sequence $(a_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ such that $\forall z \in \mathscr{C}(0, R_1, R_2), f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$.
Moreover, this series normally converges on all compact include in $\mathscr{C}(0, R_1, R_2)$.

Proof :
Let $\lambda \in \mathscr{C}(0, R_1, R_2)$.
If we consider the following map :

$$g : \left| \begin{array}{ccc} \mathscr{C}(0, R_1, R_2) & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \begin{cases} f'(\lambda) & \text{si } z = \lambda \\ \frac{f(z)-f(\lambda)}{z-\lambda} & \text{si } z \neq \lambda \end{cases} \end{array} \right.$$

The map g is continuous and its restriction on $\mathscr{C}(0, R_1, R_2) \backslash \{\lambda\}$ is holomorphic. So, we can apply the below lemma. We can set r_1, r_2 such that $R_1 < r_1 < |\lambda| < r_2 < R_2$ and so, we have

$$\int_{\Gamma_{r_2}} g(z)dz - \int_{\Gamma_{r_1}} g(z)dz = 0$$

But, $\text{Ind}(\lambda, \Gamma_{r_1}) = 0$ and $\text{Ind}(\lambda, \Gamma_{r_2}) = 1$. It means that $\frac{1}{2i\pi}\left(\int_{\Gamma_{r_2}} \frac{dz}{z-\lambda} - \int_{\Gamma_{r_1}} \frac{dz}{z-\lambda}\right) = 1$.
And, by those two equalites, we can deduce that :

$$f(\lambda) = \frac{1}{2i\pi}\left(\int_{\Gamma_{r_2}} \frac{f(z)}{z-\lambda}dz - \int_{\Gamma_{r_1}} \frac{f(z)}{z-\lambda}dz\right) \tag{1}$$

However, for $z \in \mathbb{C}$ such that $|z| = r_2 > |\lambda|$ *i.e.* $\left|\frac{\lambda}{z}\right| < 1$, we have :

$$\frac{1}{z-\lambda} = \frac{1}{z} \times \frac{1}{1-\frac{\lambda}{z}} = \frac{1}{z}\sum_{n=0}^{+\infty} \frac{\lambda^n}{z^n} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{z^{n+1}} \text{ hence, } \frac{f(z)}{z-\lambda} = \sum_{n=0}^{+\infty} \frac{f(z)}{z^{n+1}}\lambda^n$$

For the same reason, for $z \in \mathbb{C}$ such that $|z| = r_1 < |\lambda|$ *i.e.* $\left|\frac{\lambda}{z}\right| > 1$, we have :

$$\frac{1}{z-\lambda} = \frac{1}{\lambda} \times \frac{-1}{1-\frac{z}{\lambda}} = \frac{-1}{\lambda}\sum_{n=0}^{+\infty} \frac{z^n}{\lambda^n} = -\sum_{n=0}^{+\infty} z^n \lambda^{-(n+1)} = -\sum_{n=-\infty}^{-1} \frac{\lambda^n}{z^{n+1}}$$

Hence,

$$\frac{f(z)}{z-\lambda} = -\sum_{n=-\infty}^{-1} \frac{f(z)}{z^{n+1}}\lambda^n$$

So, by (1) and the below lemma (applied to $z \mapsto \frac{f(z)}{z^{n+1}}$ holomrphic on $\mathscr{C}(0, R_1, R_2)$), we have

$$f(\lambda) = \sum_{n=-\infty}^{+\infty} a_n \lambda^n \text{ where } \forall n \in \mathbb{Z}, a_n = \frac{1}{2i\pi}\int_{\Gamma_r} \frac{f(z)}{z^{n+1}} dz \text{ independantly of the choice of } r \in]R_1, R_2[.$$

As power series are normally convergent on all compacts included on their convergence disk, we can deduce that $\sum\limits_{n\in\mathbb{Z}} a_n z^n$ normally converges on all compact include on $\mathscr{C}(0, R_1, R_2)$. ■

PROPOSITION 2.1.3 :
Let $\mathscr{C}(0, R_1, R_2) \subseteq \mathbb{C}$ such that $\mathscr{C} \subseteq \mathscr{C}(0, R_1, R_2)$ and $f \in H(\mathscr{C}(0, R_1, R_2))$.
Then f has an absolutely convergent Fourier series.

PROOF :
The map f is holomorphic on the annulus, so by the below proposition, f can be expanded in Laurent series. So there exists $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ such that

$$\forall z \in \mathscr{C}(0, R_1, R_2), f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

As $\mathscr{C}$ is a compact include in $\mathscr{C}(0, R_1, R_2)$, the series $\sum\limits_{n\in\mathbb{Z}} a_n$ is normally convergent.
So, the family $(a_n)_{n\in\mathbb{Z}}$ is summable. But, we have :

$$\forall n \in \mathbb{Z}, a_n = \frac{1}{2i\pi}\oint_{\mathscr{C}} \frac{f(z)}{z^{n+1}} dz \underset{\substack{\uparrow \\ z=e^{i\theta}}}{=} \frac{1}{2i\pi}\int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{e^{i(n+1)\theta}} ie^{i\theta} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta} d\theta = c_n(f)$$

As $\sum\limits_{n\in\mathbb{Z}} a_n$ is absolutely convergent, then $\sum_{n\in\mathbb{Z}} c_n(f)$ is absolutely convergent. And so, f has an absolutely convergent Fourier series. ■

2.2 Results on $\ell^1(\mathbb{Z})$-space

Definition 2.2.1 : ($\ell^1(\mathbb{Z})$-space)
The $\ell^1(\mathbb{Z})$ **space** is defined by :

$$\ell^1(\mathbb{Z}) := \left\{ T = (t_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \text{ such that } \|T\|_1 = \sum_{k=-\infty}^{+\infty} |t_k| < +\infty \right\}$$

Proposition 2.2.2 :
The space $\ell^1(\mathbb{Z})$ is a Banach algebra in which the product of two elements is defined by convolution :

$$\forall S = (s_k)_{k\in\mathbb{Z}}, T = (t_k)_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z}),\ TS := (p_k)_{k\in\mathbb{Z}} \text{ where } \forall k \in \mathbb{Z}, p_k = \sum_{j=-\infty}^{+\infty} t_j s_{k-j}$$

Remark We can also write, for all k in $\mathbb{Z}$, $p_k = \sum\limits_{j=-\infty}^{+\infty} t_j s_{k-j} = \sum\limits_{j+l=k} t_j s_l$

Proof :

— Let us show that $(\ell^1(\mathbb{Z}), \|.\|)$ is complete. Let $(T_n)_{n\in\mathbb{Z}}$ a Cauchy sequence of $\ell^1(\mathbb{Z})$. Then :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \geq N, \|T_p - T_q\|_1 \leq \varepsilon \quad (*)$$
$$\text{i.e. } \sum_{k=-\infty}^{+\infty} |T_{p,k} - T_{q,k}| \leq \varepsilon$$

In particular, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \geq N, \forall k \in \mathbb{Z}, |T_{p,k} - T_{q,k}| \leq \varepsilon$.
So, for all k in $\mathbb{Z}$, $(T_{n,k})_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{C}, |.|)$ which is complete. Then, $(T_{n,k})_{n\in\mathbb{N}}$ converges, we denote $T_k \in \mathbb{C}$ its limit. Let $T := (T_k)_{k\in\mathbb{Z}}$.
But, $(*)$ with $\varepsilon = 1 > 0$ gives :

$$\exists N \in \mathbb{N}, \forall p, q \geq N, \|T_p - T_q\|_1 \leq 1$$

Let $K \in \mathbb{N}$. Then, $\forall p \geq N, \sum\limits_{k=-K}^{K} |T_{p,k} - T_{N,k}| \leq 1$.

So, when p goes to $+\infty$, we have : $\sum\limits_{k=-K}^{K} \underbrace{|T_k - T_{N,k}|}_{|T_k| - |T_{N,k}| \leq} \leq 1$.

So, $\sum\limits_{k=-K}^{K} |T_k| \leq \sum\limits_{k=-K}^{K} |T_{N,k}| + 1 \leq \sum\limits_{k=-\infty}^{+\infty} |T_{N,k}| + 1 = \|T_N\|_1$.
So, $\sum |T_k|$ converges *i.e.* $T \in \ell^1(\mathbb{Z})$.
Then, when q goes to $+\infty$ in $(*)$, we have :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p \geq N, \|T_p - T\|_1 \leq \varepsilon$$

This means $T_n \underset{n\to\infty}{\overset{\|\cdot\|_1}{\longrightarrow}} T$.
Hence, $(\ell^1(\mathbb{Z}), \|.\|_1)$ is a Banach space.

— Let us show that the convolution is well defined.
As $T \in \ell^1(\mathbb{Z})$, $t_k \underset{|k|\to+\infty}{\rightarrow} 0$, and then $(t_k)_{k\in\mathbb{N}}$ is bounded by a constant $M \in \mathbb{R}$.

So, for all $j \in \mathbb{Z}$, for all $k \in \mathbb{Z}, |t_j s_{k-j}| \leq M |s_{k-j}|$ and then, as $S \in \ell^1(\mathbb{Z})$, we have $(t_j s_{k-j})_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ *i.e.* $(t_j s_{k-j})_{j\in\mathbb{Z}}$ is summable.
But, we have

$$\forall j \in \mathbb{Z}, \sum_{k\in\mathbb{Z}} |t_j s_{k-j}| = |t_j| \sum_{k\in\mathbb{Z}} |s_{k-j}| = |t_j| \sum_{k\in\mathbb{Z}} |s_k|$$

So, for all $j \in \mathbb{Z}$, $\sum\limits_{k\in\mathbb{Z}} |t_j s_{k-j}|$ converges and its sum is $\sum\limits_{k=-\infty}^{+\infty} |t_j s_{k-j}| = |t_j| \left\|s\right\|_1$.
Moreover, $\sum\limits_{j\in\mathbb{Z}} \sum\limits_{k=-\infty}^{+\infty} |t_j s_{k-j}| = \left\|s\right\|_1 \sum\limits_{j\in\mathbb{Z}} |t_j|$. So, $\sum\limits_{j\in\mathbb{Z}} \sum\limits_{k=-\infty}^{+\infty} |t_j s_{k-j}|$ converges and its sum is $\sum\limits_{j=-\infty}^{+\infty} \sum\limits_{k=-\infty}^{+\infty} |t_j s_{k-j}| = \left\|s\right\|_1 \left\|t\right\|_1$.
Hence, by Fubini's theorem, $(t_j s_{k-j})_{k,j\in\mathbb{Z}}$ is summable and :

$$\sum_{(j,k)\in\mathbb{Z}^2} |t_j s_{k-j}| = \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} |t_j| \, |s_{k-j}| \right) = \left\|t\right\|_1 \left\|s\right\|_1 < +\infty$$

Then, as $\forall k \in \mathbb{Z}, \left| \sum\limits_{j=-\infty}^{+\infty} t_j s_{k-j} \right| \leq \sum\limits_{j=-\infty}^{+\infty} |t_j| \, |s_{k-j}| \, , (p_k)_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$.
So, the convolution of T and S is well defined in $\ell^1(\mathbb{Z})$ and we have :

$$\left\|TS\right\|_1 = \sum_{k=-\infty}^{+\infty} \left| \sum_{j=-\infty}^{+\infty} t_j s_{k-j} \right| \leq \sum_{k=-\infty}^{+\infty} \left(\sum_{j=-\infty}^{+\infty} |t_j| \, |s_{k-j}| \right) = \left\|T\right\|_1 \left\|S\right\|_1$$

— Let us show that the convolution is associative.
Let $R = (r_k)_{k\in\mathbb{Z}}, S = (s_k)_{k\in\mathbb{Z}}, T = (t_k)_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. Then, for all k in $\mathbb{Z}$:

$$\begin{aligned}
(R(ST))_k &= \sum_{j+l=k} r_j (ST)_l \\
&= \sum_{j+l=k} t_j \sum_{i+h=l} s_i t_h \\
&= \sum_{j+i+h=k} r_j s_i t_h \\
&= \sum_{l+h=k} \left(\sum_{j+i=l} r_j s_i \right) t_h \\
&= \sum_{l+h=k} (RS)_l t_h \\
&= ((RS)T)_k
\end{aligned}$$

Hence, $R(ST) = (RS)T$. So, the convolution is associative.

■

Proposition 2.2.3 :
The space $\ell^1(\mathbb{Z})$ is commutative.

Proof :
Let $T, S \in \ell^1(\mathbb{Z})$, $TS = (p_k)_{k\in\mathbb{Z}}$ and $ST = (q_k)_{k\in\mathbb{Z}}$. Let $k \in \mathbb{Z}$.

$$p_k = \sum_{j\in\mathbb{Z}} t_j s_{k-j} \overset{\overset{u=k-j}{\downarrow}}{=} \sum_{u\in\mathbb{Z}} t_{k-u} s_u = \sum_{u\in\mathbb{Z}} s_u t_{k-u} = q_k$$

So, for all $k \in \mathbb{N}$, $p_k = q_k$, *i.e.* $TS = ST$ ∎

Notation We will write $\mathcal{B}$ for $\ell^1(\mathbb{Z})$ in the remainder of the document

2.3 Wiener's theorem

Remark We can identify $\mathbb{R}/_{2\pi\mathbb{Z}}$ and $\mathbb{T} := \{z \in \mathbb{C}, |z| = 1\}$.
Indeed, the following map defines an isomorphism between $\mathbb{R}/_{2\pi\mathbb{Z}}$ and $\mathbb{T}$:

$$\psi : \left| \begin{array}{rcl} \mathbb{R}/_{2\pi\mathbb{Z}} & \longrightarrow & \mathbb{T} \\ \theta & \longmapsto & e^{i\theta} \end{array} \right.$$

Definition 2.3.1 : (Exponential family)
For all $k \in \mathbb{Z}$, Let us define : $e_k : \theta \in \mathbb{R} \mapsto e^{ik\theta} \in \mathbb{T}$.
We will call $(e_k)_{k\in\mathbb{Z}}$ the **exponential family.**

Definition 2.3.2 : (Fourier coefficients)
For all $f \in L^1(\mathbb{T})$, we call Fourier coefficients of f the following sequence $(\hat{f}(k))_{k\in\mathbb{Z}}$ defined by :

$$\forall k \in \mathbb{Z}, \hat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} \mathrm{d}\theta$$

Definition 2.3.3 : (Fourier series)
For all $f \in L^1(\mathbb{T})$, we can associate the following series called **Fourier series of** f :

$$\sum_{k\in\mathbb{Z}} \hat{f}(k) e_k$$

Notation For $T = (\tau_k)_{k\in\mathbb{Z}} \in \mathcal{B}$, we denote $\Delta_T := \sum_{k=-\infty}^{\infty} \tau_k e_k$.

Proposition 2.3.4 :
Let $T = (\tau_k)_{k\in\mathbb{Z}} \in \mathcal{B}$. For all $\theta \in \mathbb{R}$, $\Delta_T(\theta)$ is an absolutely convergent series.

Proof :
Let $\theta \in \mathbb{R}$. As $T \in \mathcal{B}$,

$$\sum_{k=-\infty}^{\infty} |\tau_k e_k(\theta)| = \sum_{k=-\infty}^{\infty} |\tau_k| \underbrace{|e_k(\theta)|}_{=1} < +\infty$$

∎

Proposition 2.3.5 :
Let $T = (\tau_k)_{k\in\mathbb{Z}}, S = (s_k)_{k\in\mathbb{Z}} \in \mathcal{B}$.
Then, for all $\theta \in \mathbb{R}$, $\Delta_{TS}(\theta) = \Delta_T(\theta)\Delta_S(\theta)$.

PROOF :
Let $\theta \in \mathbb{R}$.
As $T, S \in \mathcal{B}, TS := (p_k)_{k\in\mathbb{Z}} \in \mathcal{B}$, the two series $\Delta_T(\theta)$ and $\Delta_S(\theta)$ are absolutely convergent. So the Cauchy product of the two series converges.So, we have :

$$\begin{aligned}\Delta_T(\theta)\Delta_S(\theta) &= \left(\sum_{k=-\infty}^{\infty} \tau_k e_k(\theta)\right)\left(\sum_{k=-\infty}^{\infty} s_k e_k(\theta)\right)\\ &= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \tau_j e_j(\theta) s_{k-j} e_{k-j}(\theta)\\ &= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \tau_j s_{k-j} e_k(\theta)\\ &= \sum_{k=-\infty}^{\infty} p_k e_k(\theta)\\ &= \Delta_{TS}(\theta)\end{aligned}$$

■

PROPOSITION 2.3.6 :
Let $T = (\tau_k)_{k\in\mathbb{Z}} \in \mathcal{B}$. Then, $\sigma(T) = \Delta_T(\mathbb{R})$.

PROOF :

$i)$ Let us suppose that Δ_T has an analytic continuation on a neighbourhood of $\mathscr{C}$ that we denote V. Let us denote f this continuation.
Let $\zeta \notin \Delta_T(\mathbb{R})$.
Then, as f is continuous, there exists $V' \subseteq V$ such that :

$$\mathscr{C} \subseteq V' \text{ and } \forall z \in V', f(z) - \zeta \neq 0 \textit{ i.e. } (f-\zeta)(z) \neq 0$$

Hence, $g : z \in V' \mapsto (f(z) - \zeta)^{-1}$ is well defined and holomorphic. So, by proposition 2.1.3, g has an absolutely convergent Fourier series.
Let us denote $R := (\hat{g}(n))_{n\in\mathbb{Z}}$. Then, we have $R \in \mathcal{B}$ and $R(T-\zeta) = (T-\zeta)R = 1$. Hence, $T - \zeta$ is invertible. Then, $\zeta \notin \sigma(T)$.
So we showed $\sigma(T) \subseteq \Delta_T(\mathbb{R})$.
Converserly, if $\zeta \in \Delta_T(\mathbb{R})$ there exists $z \in \mathbb{R}$ such that $\zeta = \Delta_T(z)$ *i.e.* $\Delta_T(z) - \zeta = 0$ then $T - \zeta$ is not invertible. So, $\Delta_T(\mathbb{R}) \subseteq \sigma(T)$.

$ii)$ Now, we consider the general case.
For all $n \in \mathbb{N}$, let $T_n := (\tau_{n,k})_{k\in\mathbb{Z}}$ where

$$\tau_{n,k} = \tau_k \text{ if } k \in [\![-n, n]\!] \\ 0 \text{ else}$$

Then, as trignometric polynomials, the T_n have an analytic continuation on a neighbourhood of $\mathscr{C}$. So, we can apply the first point to them. Moreover :

$$\begin{aligned}\|T-T_n\| &= \left\| \sum_{k\leq -n-1} \tau_k + \sum_{k\geq n+1} \tau_k \right\| \\ &\leq \left\| \sum_{k\leq -(n+1)} \tau_k \right\| + \left\| \sum_{k\geq n+1} \tau_k \right\| \\ &\leq \underbrace{\sum_{k\leq -(n+1)} |\tau_k|}_{\text{rest of a convergent series}} + \underbrace{\sum_{k\geq n+1} |\tau_k|}_{\text{rest of a convergent series}}\end{aligned}$$

Hence, $\|T-T_n\| \underset{n\to\infty}{\longrightarrow} 0$.

Moreover, with a similar proof, we can show that :

$$\forall \theta \in \mathbb{R}, |\Delta_T(\theta) - \Delta_{T_n}(\theta)| \leq \sum_{k\leq -(n+1)} |\tau_k| + \sum_{k\geq n+1} |\tau_k| \underset{n\to+\infty}{\longrightarrow} 0 \text{ independantly of } \theta$$

So, $(\Delta_{T_n})_{n\in\mathbb{N}}$ uniformly converges on $\mathbb{R}$ towards Δ_T.
Then, by corollary 1.5.3,

$$\sigma(T) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \sigma(T_k)} \overset{\substack{\text{first point} \\ \downarrow}}{=} \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \Delta_{T_k}(\mathbb{R})}$$

Let us show that $\bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \Delta_{T_k}(\mathbb{R})} = \Delta_T(\mathbb{R})$:

- Let $y \in \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \Delta_{T_k}(\mathbb{R})}$.
  Then, $\forall n \in \mathbb{N}, y \in \overline{\bigcup_{k\geq n} \Delta_{T_k}(\mathbb{R})}$.
  So, let $n \in \mathbb{N}^*, \exists k_n \geq n, \exists x_n \in \mathbb{R}$ such that $|y - \Delta_{T_{k}}(x_n)| \leq \frac{1}{n}$.
  As the maps Δ_{T_k} are 2π-periodic, we can suppose that $(x_n)_{n\in\mathbb{N}^*} \subseteq [0, 2\pi]$. So, $(x_n)_{n\in\mathbb{N}^*}$ is a sequence in a compact set so it admits a convergent subsequence : there exist $\phi : \mathbb{N}^* \mapsto \mathbb{N}^*$ an increasing map and x in $[0, 2\pi]$ such that $x_{\phi(n)} \underset{n\to\infty}{\longrightarrow} x$.
  Hence, as $(\Delta_{T_k})_{k\in\mathbb{N}}$ converges uniformely towards Δ_T on $[0, 2\pi]$, we have :

$$y = \lim_{n\to\infty} \Delta_{T_{k_{\phi(n)}}}(x_{\phi(n)}) = \Delta_T(x) \in \Delta_T(\mathbb{R})$$

- Let $y \in \Delta_T(\mathbb{R})$. Then, there exists $x \in \mathbb{R}$ such that

$$\begin{aligned} y = \Delta_T(x) &= \sum_{k=-\infty}^{+\infty} \tau_k e^{ikx} \\ &= \lim_{N\to+\infty} \sum_{k=-N}^{N} \tau_k e^{ikx} \\ &= \lim_{N\to+\infty} \Delta_{T_N}(x) \\ &\in \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \Delta_{T_k}(\mathbb{R})} \end{aligned}$$

  So, $\Delta_T(\mathbb{R}) \subseteq \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \Delta_{T_k}(\mathbb{R})}$.

Then we showed that

$$\sigma(T) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \Delta_{T_k}(\mathbb{R})} = \Delta_T(\mathbb{R})$$

■

THEOREM 2.3.7 : WIENER'S THEOREM

Let $f : \mathbb{R} \mapsto \mathbb{C}$ be a 2π-periodic continuous map.
If f has an absolutely convergent Fourier series and does not vanish anywhere, then f^{-1} has an absolutely convergent Fourier series.

PROOF :
As f has an absolutely convergent Fourier series, $T := (\hat{f}(n))_{n\in\mathbb{Z}} \in \mathcal{B}$ and we have $f = \Delta_T$ almost everywhere because f is continuous. Hence, thanks to proposition 2.3.6, we have :

$$\sigma(T) = \Delta_T(\mathbb{R}) = f(\mathbb{R})$$

So, since f does not vanish anywhere, $0 \notin f(\mathbb{R}) = \sigma(T)$. Then, $T - 0 = T$ is invertible. Let us denote S its inverse. Then, by proposition 2.3.5,

$$f\Delta_S = \Delta_T\Delta_S = \Delta_{TT^{-1}} = \Delta_{1_{\mathcal{B}}} = e_0 = 1$$

As the usual product on maps is commutative, f is invertible and $f^{-1} = \Delta_S$.
Hence, f^{-1} has an absolutely convergent Fourier series. ■